

# SU( $N$ ) Fermi liquid at finite temperature

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We consider the thermodynamic potential  $\Omega$  of an  $N$  component Fermi gas with a short range interaction obeying SU( $N$ ) symmetry. We analyze especially the non-analytic part of  $\Omega$  in the temperature  $T$  at low  $T$ . We examine the temperature range where one can observe this  $T^4 \ln T$  contribution and discuss how it can be extracted experimentally.

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## I. INTRODUCTION

For a non-interacting Fermi gas, textbooks [1, 2] teach us that the specific heat at low temperatures  $T$  is a power series involving only odd powers of  $T$ , as can easily be shown by the Sommerfeld expansion. Correspondingly, the grand thermodynamic potential  $\Omega$  as a function of the temperature  $T$ , is thus a power series in  $T$  with only even powers. Interestingly, this result is qualitatively modified for an interacting Fermi system, even with short range interactions in two or three dimensions. Though the first term in the expansion for the specific heat in  $T$  indeed starts with  $T$ , (for  $\Omega$ , a constant and a  $T^2$  term), the next term is now widely believed (see references below) to be of the form  $T^3 \ln T$  (correspondingly for  $\Omega$ ,  $T^4 \ln T$ ) for three dimensions, thus not even analytic in  $T$ . Thoroughly understanding these non-analytic terms is crucial in order to distinguish them from those arising in non-Fermi liquid phases or near quantum critical points [3].

Historically, the study of this  $T^3 \ln T$  term was motivated by the experimental observation in normal liquid  $^3\text{He}$  that the specific heat cannot be fitted by the Sommerfeld expression [4]. On the theory side, this non-analytic term can be understood to be due to the presence of bosonic excitations (interacting particle-hole pairs) in the system, even though these excitations are not necessarily propagating but can be overdamped (see, e.g. Refs. [5, 6]). More interestingly, for the  $T^3 \ln T$  term at least, the result can be obtained via a proper extension of the original Landau Fermi liquid theory [7–11], and the coefficient of this  $T^3 \ln T$  term can be entirely expressed in terms of scattering amplitudes between Landau quasiparticles on the Fermi surface [12, 13]. The non-analytic behavior in specific heat or thermodynamic potential is a consequence of the non-analytic behavior of the density and spin susceptibilities of the system at finite frequencies and wavevectors [14].

The theories of Refs. [12–14] yield results for this term that are in reasonable agreement with experiments [4] in  $^3\text{He}$ . However, precise statements are difficult to make due to some uncertainties in the interacting parameters in this system [12–14]. This  $T^3 \ln T$  term has also been studied in heavy fermion materials such as UPt<sub>3</sub> and UAl<sub>2</sub>

[15, 16]. However, there the interaction parameters are even less known than those in  $^3\text{He}$ . Therefore it is highly desirable to have another system where these theories can be tested.

In this paper, we analyze the thermodynamic potential of interacting SU( $N$ ) Fermi gases such as  $^{173}\text{Yb}$  and  $^{87}\text{Sr}$  which are available now in cold atom experiments [17–23] (see also the review [24]). These  $N_c$  components (we use  $N_c$  rather than  $N$  here to avoid possible confusion with the number of particles.) represent the different choices of hyperfine spin sublevels  $m_f$  available to the atoms.  $m_f = -5/2, \dots, 5/2$  for  $^{173}\text{Yb}$  and  $m_f = -9/2, \dots, 9/2$  for  $^{87}\text{Sr}$ . We would like to in particular examine whether these systems can serve as candidates to test these theories. Both the number of components  $N_c$ , ( $N_c$  can vary from 1 to 6 in  $^{173}\text{Yb}$  and 1 to 10 in  $^{87}\text{Sr}$ ) and the density (hence the dimensionless coupling constant defined below) can be varied in experiments. (The former is possible due to the SU( $N$ ) symmetry of the interparticle interaction [22, 23].) For a sufficiently large cloud of the gas where the local density approximation can be taken, the pressure  $P$  of the gas (which is equal to  $-\Omega$  per unit volume) can be deduced from the axial density [25]. Since the effective chemical potential varies across the trap, analysis of this data can then produce the grand thermodynamic potential  $\Omega$  as a function of the chemical potential  $\mu$ . If the temperature can also be measured, then the function  $\Omega(\mu, T)$  can be obtained and compared with theory. These types of studies have already been carried out extensively for many systems, including two-component resonant Fermi gases [26–30], one-component interacting Bose gas [31], and we expect that the same can be done for the  $^{173}\text{Yb}$  and  $^{87}\text{Sr}$  systems eventually. Previously we have investigated theoretically the Fermi liquid properties of this SU( $N$ ) Fermi gas at zero temperature [32], and we here extend our study to finite temperatures, limiting ourselves to three dimensions in this paper. While the theories in Refs. [12–14] pointed out the existence of a  $T^3 \ln T$  term in the specific heat and thus a  $T^4 \ln T$  term in  $\Omega(\mu, T)$ , these calculations have not been verified numerically to the best of our knowledge. More importantly, they also offer us no hint on the temperature range where one can find such non-analytic behavior. We here

evaluate the contributions to  $\Omega(\mu, T)$  term by term numerically at arbitrary temperatures which then allow us to answer this question.

In principle the non-analytic terms in the thermodynamic potential can also be investigated for the resonant two-component system or multi-component Fermi system without  $SU(N)$  symmetry, but we shall discuss how the variable  $N_c$  may offer us some advantage.

## II. THE THERMODYNAMIC POTENTIAL $\Omega(\mu, T)$

We present here the evaluation of the thermodynamic potential  $\Omega(\mu, T)$  of a  $N_c$  component interacting Fermi

gas as a function of the chemical potential  $\mu$  and temperature  $T$ .  $\mu$  is taken to be the same for all  $N_c$  components. The interparticle interaction is characterized by a positive s-wave scattering length  $a$ , which is the same irrespective of the hyperfine spin sublevels  $m_f$ 's of the fermions participating in the interaction [22, 23]. Note that the pressure  $P$  is just  $-\Omega/V$ , where  $V$  is the volume. We evaluate  $\Omega(\mu, T)$  up to second order in  $a$ , expressed as a power series in the dimensionless parameter  $k_\mu a$ , where  $k_\mu \equiv (2M\mu)^{1/2}$  with  $M$  the mass of an atom.  $k_\mu$  would be equal to the Fermi momentum in the special limit of zero temperature and in the absence of interactions.

The zeroth order term of  $\Omega(\mu, T)$  in  $a$  is simply that of the free gas,  $\Omega_0(\mu, T)$ , and hence given by

$$\Omega_0(\mu, T) = N_c \sum_{\vec{k}} [(\epsilon_k^0 - \mu) f_k^0 + T(f_k^0 \ln f_k^0 + (1 - f_k^0) \ln(1 - f_k^0))]$$

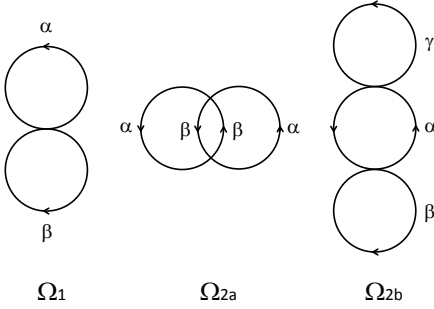


FIG. 1: Feynman diagrams for the first  $\Omega_1$  and second order contributions  $\Omega_{2a}$ ,  $\Omega_{2b}$  in interaction to the thermodynamic potential.

where  $\epsilon_k^0 \equiv \frac{k^2}{2M}$  is just the kinetic energy,

$$f_k^0(\mu, T) \equiv \frac{1}{\exp(\frac{\epsilon_k^0 - \mu}{T}) + 1}$$

the Fermi function, both of a free particle of wavevector  $\vec{k}$ .  $k$  is the magnitude of  $\vec{k}$ . Here we have already used the fact that  $f_{\vec{k},\alpha}^0, f_{\vec{k},\beta}^0, \dots$ , the distribution functions for

species  $\alpha, \beta, \dots$  are all given by  $f_k^0(\mu, T)$ , since we have assumed equal chemical potentials for all species. We have thus

$$\Omega_0(\mu, T) = V N_c T \int \frac{d^3 k}{(2\pi)^3} \ln(1 - f_k^0) \quad (1)$$

The Feynmann diagrams for the first and second order terms in  $a$  are shown in Fig 1. The first order term  $\Omega_1(\mu, T)$  is simply  $\frac{4\pi a}{MV} \sum_{\alpha > \beta} N_\alpha^0 N_\beta^0$ , where  $N_\alpha^0$ ,  $N_\beta^0$  are the total number of particles for species  $\alpha$  and  $\beta$  respectively. In this expression, the sum over components is restricted to different species since, for short range interactions, the contributions from direct and exchange interaction cancel for identical species. Furthermore, we have used the fact that, at this order, it is sufficient to use the particle numbers  $N_{\alpha,\beta}^0$  at zeroth order. We thus have

$$\Omega_1(\mu, T) = V \frac{N_c(N_c - 1)}{2} \frac{4\pi a}{M} n_\alpha^0 n_\beta^0 \quad (2)$$

where there is no sum over  $\alpha, \beta$  in the above formula, and  $n_\alpha^0(\mu, T)$  is simply the number density of the  $\alpha$  component, given by  $\int \frac{d^3 k}{(2\pi)^3} f_k^0$ .

There are two diagrams to second order in  $a$ . The first one, which we shall denote as  $\Omega_{2a}$  and is depicted in the middle of Fig. 1, is given by

$$\Omega_{2a}(\mu, T) = -\frac{N_c(N_c - 1)}{2} \left( \frac{4\pi a}{MV} \right)^2 \sum_{\vec{k}_1, \vec{k}_2, \vec{k}_1'} \frac{f_{\vec{k}_1, \alpha}^0 f_{\vec{k}_2, \beta}^0 (f_{\vec{k}_1', \alpha}^0 + f_{\vec{k}_2', \beta}^0)}{\frac{k_1^2 + k_2^2 - k_1'^2 - k_2'^2}{2M}} \quad (3)$$

where  $\alpha, \beta$  again are not summed, and  $\vec{k}'_2 \equiv \vec{k}_1 + \vec{k}_2 - \vec{k}'_1$ .  $\Omega_{2a}$  is the only term in the thermodynamic potential, up to this order in  $a$ , which is responsible for modifications of the physical properties of the system that cannot be regarded as just a chemical potential shift due to interaction. This diagram is also responsible for the induced interaction among Landau quasiparticles studied in, e.g., Refs. [32, 33]. The second diagram, which we shall denote as  $\Omega_{2b}$  and is depicted on the right part of Fig. 1, can be considered as a Hartree correction to the diagram for  $\Omega_1$ : for example, one can regard the line labeled by  $\gamma$  is simply giving a constant energy shift  $\delta\epsilon = \frac{4\pi a}{M} n_\gamma^0$  to the propagator labeled by  $\alpha$ . Noting the combinatorial factor of  $1/2!$  for second order interaction diagrams, the part that is of order  $a^2$  is thus  $\frac{1}{2} \frac{4\pi a}{MV} \sum_{\alpha>\beta} (N_\alpha - N_\alpha^0) N_\beta^0$  where the difference  $(N_\alpha - N_\alpha^0)/V$  is given by

$$\int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{\exp(\frac{\epsilon_k^0 + \delta\epsilon - \mu}{T}) + 1} - \frac{1}{\exp(\frac{\epsilon_k^0 - \mu}{T}) + 1} \right) = -\frac{\partial n_\alpha^0}{\partial \mu}(\delta\epsilon) \quad .$$

Summing over possible choices of  $\gamma$  finally gives us

$$\Omega_{2b}(\mu, T) = V \frac{N_c(N_c - 1)^2}{2} \left( \frac{4\pi a}{M} \right)^2 \left( -\frac{\partial n_\alpha^0}{\partial \mu} \right) n_\beta^0 n_\gamma^0 \quad (4)$$

where again  $\alpha, \beta, \gamma$  are not summed.

We therefore have, up to second order in  $a$ ,

$$\Omega(\mu, T) = V N_c \frac{k_\mu^3}{6\pi^2} \frac{k_\mu^2}{2M} \{ \tilde{\omega}_0 + (k_\mu a) \tilde{\omega}_1 + (k_\mu a)^2 (\tilde{\omega}_{2a} + \tilde{\omega}_{2b}) \} \quad , \quad (5)$$

with

$$\begin{aligned} \tilde{\omega}_0 &= \omega_0 \\ \tilde{\omega}_1 &= (N_c - 1) \omega_1 \\ \tilde{\omega}_{2a} &= (N_c - 1) \omega_{2a} \\ \tilde{\omega}_{2b} &= (N_c - 1)^2 \omega_{2b} \end{aligned} \quad (6)$$

where  $\omega_0, \dots, \omega_{2b}$  are  $N_c$  independent dimensionless functions of  $\mu, T$  and hence only of  $t \equiv T/\mu$ , and  $\omega_0$  originated from  $\Omega_0$ ,  $\omega_1$  from  $\Omega_1$ , etc. We shall provide the explicit expressions for  $\omega_0, \dots, \omega_{2b}$  later after we discuss the zero temperature limit.

At  $T = 0$ , we easily find, using  $n_\alpha(\mu, 0) = k_\mu^3/6\pi^2$ ,  $\partial n_\alpha(\mu, 0)/\partial \mu = Mk_\mu/2\pi^2$ ,

$$\omega_0(0) = -\frac{2}{5} \quad , \quad (7)$$

and

$$\omega_1(0) = \frac{2}{3\pi} \quad . \quad (8)$$

$\omega_{2a}$  is given by a rather complicated integral but has already been evaluated before in the literature, as the same integral appears in the energy of a two-component Fermi gas to second order in  $a$ , see for example §6 of Ref. [9]. We find then

$$\omega_{2a}(0) = \frac{4}{35} \frac{(11 - 2 \ln 2)}{\pi^2} \approx 0.11132 \quad . \quad (9)$$

$\omega_{2b}$  can be easily found to be

$$\omega_{2b}(0) = -\frac{4}{3\pi^2} \approx -0.1351 \quad . \quad (10)$$

The total number of particles  $N_{\text{tot}}$  can be found via  $-\partial\Omega/\partial\mu$ , and defining the Fermi momentum  $k_F$  via  $N_{\text{tot}} = V N_c \frac{k_F^3}{6\pi^2}$  (with  $N_{\text{tot}}$  the total number of particles at  $T = 0$ ) gives us

$$\begin{aligned} k_F &= k_\mu \left\{ 1 - 3(k_\mu a) \tilde{\omega}_1 - \frac{7}{2} (k_\mu a)^2 \tilde{\omega}_2 \right\}^{1/3} \\ &\approx k_\mu \left\{ 1 - (k_\mu a) \tilde{\omega}_1 - (k_\mu a)^2 \left[ \frac{7}{6} \tilde{\omega}_2 + \tilde{\omega}_1^2 \right] \right\} \end{aligned} \quad (11)$$

where  $\tilde{\omega}_2 = \tilde{\omega}_{2a} + \tilde{\omega}_{2b}$ . Using the relation  $E = \Omega + \mu N_{\text{tot}}$  for the total energy  $E$  at zero temperature and eliminating  $\mu$  in favor of  $k_F$ , one can check that (see Appendix A) our expressions above reproduce the result for  $E$  given in the literature (e.g. Refs. [9, 34, 35]).

At finite temperatures, the dimensionless functions  $\omega_0, \dots, \omega_{2b}$  are given by

$$\omega_0(t) = 3 \frac{T}{\mu} \frac{1}{k_\mu^3} \int_0^\infty dk k^2 \ln(1 - f_k^0) \quad , \quad (12)$$

$$\omega_1(t) = \frac{2}{3\pi} \left( \frac{n_\alpha^0(\mu, T)}{n_\alpha^0(\mu, 0)} \right)^2 \quad , \quad (13)$$

$$\omega_{2a}(t) = -\frac{3 \times 2^6 \times \pi^4}{k_\mu^7} \int_{\vec{k}_1} \int_{\vec{k}_2} \int_{\vec{k}'_1} \frac{f_{\vec{k}_1} f_{\vec{k}_2} (f_{\vec{k}'_1} + f_{\vec{k}'_2})}{k_1^2 + k_2^2 - k_1'^2 - k_2'^2} \quad (14)$$

(where we have introduced the short-hand  $f_{\vec{k}} \equiv f(\frac{d^3k}{(2\pi)^3})$ , and

$$\omega_{2b}(t) = -\frac{4}{3\pi^2} \left( \frac{n_\alpha^0(\mu, T)}{n_\alpha^0(\mu, 0)} \right)^2 \left( \frac{\partial n_\alpha^0(\mu, T)/\partial \mu}{\partial n_\alpha^0(\mu, 0)/\partial \mu} \right) \quad . \quad (15)$$

Below we discuss the low temperature expansion of  $\Omega(\mu, T)$ . It is convenient first to ignore the contribution from  $\omega_{2a}$ , that is, we include the Hartree-Fock diagrams only. We shall call this result  $\Omega^{\text{HF}}(\mu, T)$ . The low temperature formulas for  $\omega_{0,1,2b}$  can be easily obtained by standard Sommerfeld expansion. We get

$$\omega_0(t) = -\frac{2}{5} \left[ 1 + \frac{5\pi^2}{8}t^2 - \frac{7\pi^4}{384}t^4 + \dots \right], \quad (16)$$

and with the help of

$$\frac{n_\alpha^0(\mu, T)}{n_\alpha^0(\mu, 0)} = 1 + \frac{\pi^2}{8}t^2 + \frac{7\pi^4}{640}t^4 + \dots,$$

$$\frac{\partial n_\alpha^0(\mu, T)/\partial \mu}{\partial n_\alpha^0(\mu, 0)/\partial \mu} = 1 - \frac{\pi^2}{24}t^2 - \frac{7\pi^4}{384}t^4 + \dots,$$

also

$$\omega_1(t) = \frac{2}{3\pi} \left[ 1 + \frac{\pi^2}{4}t^2 + \frac{3\pi^4}{80}t^4 + \dots \right] \quad (17)$$

and

$$\omega_{2b}(t) = -\frac{4}{3\pi^2} \left[ 1 + \frac{5\pi^2}{24}t^2 + \frac{17\pi^4}{1920}t^4 + \dots \right] \quad (18)$$

where  $t \equiv T/\mu$  and the higher order terms not displayed here are  $t^6$ ,  $t^8$  etc.

The above implies, to order  $T^2$ ,

$$\Omega^{\text{HF}}(\mu, T) = \Omega^{\text{HF}}(\mu, 0) - VN_c \frac{Mk_\mu}{2\pi^2} \frac{\pi^2 T^2}{6} \left[ 1 - (N_c - 1) \frac{2k_\mu a}{3\pi} + (N_c - 1)^2 \frac{10}{9} \left( \frac{k_\mu a}{\pi} \right)^2 \right]. \quad (19)$$

This result is in accordance with the expectation from Fermi liquid theory [9, 10], though with interactions now restricted to Hartree-Fock. In this theory, the specific heat should be linear in  $T$  at low temperatures, and is given by  $VN_c \frac{Mk_F^{\text{HF}}}{2\pi^2} \frac{\pi^2 T}{3}$ . Note that the density of states for each fermion component  $\frac{Mk_F^{\text{HF}}}{2\pi^2}$  that enters here is related to the Fermi wavevector  $k_F^{\text{HF}}$  for the corresponding particle density *at zero temperature*. Since we are using the chemical potential as an independent variable,  $k_F^{\text{HF}}$  is given by the corresponding Hartree-Fock value, thus by Eq. (11) but with the contribution from  $\tilde{\omega}_{2a}$  dropped. Indeed, using Eqs. (7),(8) and (10), we obtain

$$k_F^{\text{HF}} = k_\mu \left[ 1 - (N_c - 1) \frac{2k_\mu a}{3\pi} + (N_c - 1)^2 \frac{10}{9} \left( \frac{k_\mu a}{\pi} \right)^2 \right]. \quad (20)$$

Together with  $S^{\text{HF}} = -\partial\Omega^{\text{HF}}/\partial T$ , and noting that to linear order in  $T$ , the entropy  $S^{\text{HF}}(\mu, T)$  of the system is given by the same expression as the specific heat, we verify our claim above.

$\omega_{2a}$ , in contrast to the other terms discussed above, is *not* expected to be analytic in  $t$ . Rather, we anticipate

$$\omega_{2a}(t) = \frac{4}{35\pi^2} (11 - 2\ln 2) + B_{22}t^2 + B_{23}t^4 \ln t + B_{24}t^4 + \dots \quad (21)$$

The first term was already given in Eq. (9).

Let us first discuss  $B_{22}$ . With similar discussions for the Hartree-Fock contributions above, we expect from Fermi liquid theory that the thermodynamic potential,

up to  $T^2$ , is given by

$$\Omega(\mu, T) = \Omega(\mu, 0) - VN_c \frac{M^*k_F}{2\pi^2} \frac{\pi^2 T^2}{6} \quad (22)$$

with now  $k_F$  given by Eq. (11), and  $M^*$  the effective mass of the quasiparticles.  $M^*$  is available from standard text with rather straight-forward extension [32] to our  $\text{SU}(N)$  system. We have

$$M^*/M = 1 + (N_c - 1) \frac{8}{15\pi^2} (7\ln 2 - 1)(k_\mu a)^2 \quad (23)$$

where we have already taken the liberty that, at this order, we can simply use  $k_\mu$  instead of  $k_F$  in the last term. Eqs. (22) and (19) imply that we expect

$$\Omega_{2a}(\mu, T) = \Omega_{2a}(\mu, 0) - VN_c \frac{M^*k_F - Mk_F^{\text{HF}}}{2\pi^2} \frac{\pi^2 T^2}{6}. \quad (24)$$

Using Eqs. (23) and (11), we obtain, to second order in  $a$ ,

$$M^*k_F - Mk_F^{\text{HF}} = Mk_\mu(N_c - 1) \left( \frac{k_\mu a}{\pi} \right)^2 2[2\ln 2 - 1].$$

Eq. (5) together with Eq. (21) show that we anticipate

$$B_{22} = -\ln 2 + 0.5 \approx -0.1936, \quad (25)$$

a value which we shall verify independently below.

The term proportional to  $B_{23}$  is the first non-analytic contribution to  $\Omega_{2a}(\mu, T)$  and hence  $\Omega(\mu, T)$  at low temperatures. Theories presented in Refs. [12, 14] provided

formulas for this quantity, and their results are in agreement with each other. In Ref. [12], Eq. (22), the non-analytic contribution to the entropy  $S$  was written as, for the two-component system  $N_c = 2$ ,

$$\Delta S = -V \frac{\pi^4}{20} n_{\text{tot}} B^s \left( \frac{T}{T_F} \right)^3 \ln \left( \frac{T}{T_F} \right) \quad (26)$$

where  $n_{\text{tot}}$  is the total density,  $T_F$  the Fermi temperature. To our required accuracies we can put  $n_{\text{tot}} = 2k_\mu^3/6\pi^2$  ( $N_c = 2$ ), and replace  $T_F$  by  $\mu$ .  $B^s$  is a quantity that can be expressed in terms of scattering amplitudes between particles. To second order in these amplitudes, we have, via Eq. (65) in Ref. [12],

$$B^s = -\frac{1}{2} [(A_0^s)^2 + 3(A_0^a)^2] \quad (27)$$

where  $A_0^s, A_0^a$  are the angular-averaged scattering amplitudes symmetric and antisymmetric respectively with respect to spins. To lowest order in  $a$ , they are in turn given by  $A_0^s = -A_0^a = \frac{2k_\mu a}{\pi}$ . The same  $\Delta S$  can be obtained from Ref. [14] by combining their Eqs. (39), (11) and (12) with their  $U$  replaced by  $\frac{4\pi a}{M}$ . On the other hand,  $\Delta S$  can be obtained from  $-\partial\Omega/\partial T$ , noting that it originates the  $B_{23}$  term of Eq. (21) of  $\omega_{2a}$  only. We get

$$\Delta S = -VN_c(N_c - 1) \frac{k_\mu^3}{6\pi^2} (k_\mu a)^2 (4B_{23}) t^3 \ln t. \quad (28)$$

Comparison between Eqs. (26) and (28) gives

$$B_{23} = -\frac{\pi^2}{10} \approx -0.9869, \quad (29)$$

a value which we shall check also later.

Now we present our numerical results. We first consider  $\omega_{2a}(t)$ , presented in the inset of Fig. 2. Our numerical results for this quantity at low temperatures agree with what we expect from Eq. (21) with  $B_{22}$  given in Eq. (25). Subtracting these lower order (constant and  $t^2$ ) analytic terms and defining the resultant quantity to be  $\delta\omega_{2a}(t)$ , the plot of  $\delta\omega_{2a}/t^4$  as a function of  $\ln t$  is given in the main part of Fig. 2. The lower temperature data show clearly a  $t^4 \ln t$  contribution to  $\omega_{2a}(t)$ , applicable for  $t$  up to  $\approx 0.2$ , where then we find deviation from Eq. (21) due to contributions from higher order terms in  $t$  (which likely also contain further non-analytic contributions). The slope of this plot gives  $B_{23}$  also in good agreement Eq. (29). There are some deviations from the straight-line for very low temperatures due to numerical inaccuracies from the subtraction. The fit also gives us  $B_{24} \approx 1.62$ .

Fig. 3 shows an example for the total thermodynamic potential  $\Omega(\mu, T)$ , in units of  $VN_c k_\mu^5/(12\pi^2 M)$  (see Eq. (5)) for various values of  $N_c$ . The non-analytic contributions are not directly evident from this plot. The inset shows the behavior of the analytic contributions  $\omega_{0,1,2b}(t)$  plotted in analogous manner to the main Fig. 2, that is, after subtraction of the lower order  $t$  terms and divided

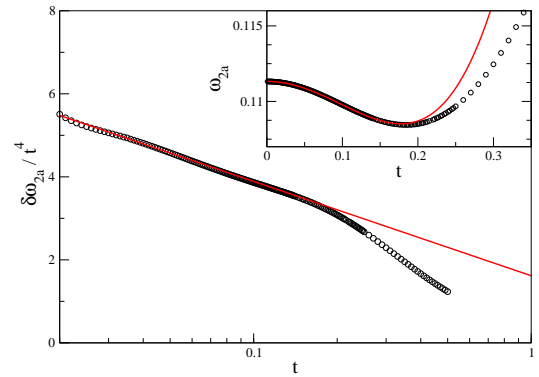


FIG. 2: (Color online) Inset:  $\omega_{2a}(t)$  as a function of reduced temperature  $t \equiv T/\mu$ . Main figure:  $\delta\omega_{2a}(t)$  divided by  $t^4$ , plotted as a function of  $\ln t$ . Here  $\delta\omega_{2a}(t) \equiv \omega_{2a} - B_{20} - B_{22}t^2$ , that is,  $\omega_{2a}(t)$  after subtraction of the lower order analytic terms in  $t$ . The values of  $B_{20}$  and  $B_{22}$  used in this subtraction are 0.1113 and  $-0.1936$ . The straight line corresponds to  $B_{23} = -0.9869$ . The fit also gives  $B_{24}$  defined in Eq. (21) to be approximately 1.62.

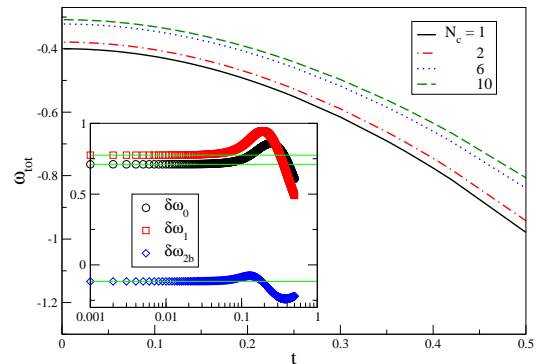


FIG. 3: (Color online)  $\omega_{\text{tot}} \equiv \{\Omega(\mu, T)/[VN_c k_\mu^5/(12\pi^2 M)]\}$  as a function of reduced temperature  $t \equiv T/\mu$ , with  $k_\mu a = 0.1$ . Inset:  $\delta\omega_{0,1,2b}(t)/t^4$  as functions of  $\ln t$ .  $\delta\omega_{0,1,2b}(t)$  are defined as  $\omega_{0,1,2b}(t)$  after subtraction of their lower order (constants and  $t^2$ ) analytic terms (c.f. Fig 2).

by  $t^4$ . Since these quantities are power series in  $t^2$ , after these subtractions they become roughly constants at low temperatures in this plot. Their intersections with the  $y$  axis give values that are in full agreement with the  $t^4$  coefficients in Eqs. (16-18). Deviations from the horizontal lines are due to contributions from higher order ( $t^6, t^8, \dots$ ) terms. We see that they become significant for  $t \gtrsim 0.1$ .

If experimentally the pressure and hence  $\Omega(\mu, T)$  can be measured for various  $\mu$ ,  $T$  and  $N_c$ , one can normalize this quantity to  $VN_c k_\mu^5/(12\pi^2 M)$ , extract the coefficients of  $k_\mu a$  and  $(k_\mu a)^2$  and obtain the quantities  $\tilde{\omega}_{0,1}$  and  $\tilde{\omega}_{2a} + \tilde{\omega}_{2b}$  defined in Eq. (5). One can then fit  $\tilde{\omega}_{2a}(t) + \tilde{\omega}_{2b}(t)$  at low temperatures to obtain the  $t = 0$

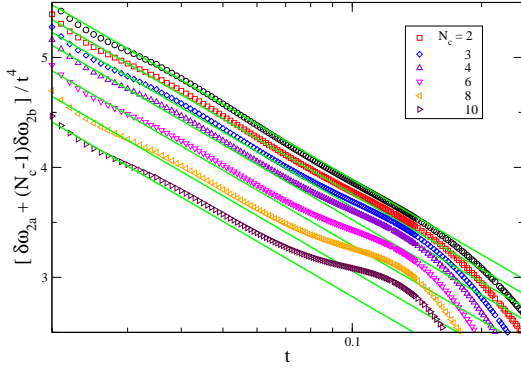


FIG. 4: (Color online)  $[\delta\tilde{\omega}_{2a}(t) + \delta\tilde{\omega}_{2b}(t)]/(N_c - 1)t^4 = [\delta\omega_{2a}(t) + (N_c - 1)\delta\omega_{2b}(t)]/t^4$  as a function of  $\ln t$ .  $\delta\tilde{\omega}_{2a}(t) + \delta\tilde{\omega}_{2b}(t)$  is defined as  $\tilde{\omega}_{2a}(t) + \tilde{\omega}_{2b}(t)$  after subtraction of their lower order analytic terms. Circles  $\circ$  represent the limit  $N_c \rightarrow 1$  hence  $\delta\omega_{2a}(t)/t^4$ .

value and a  $t^2$  contribution. Subtracting these lower order analytic terms and let us define the resulting quantity to be  $\delta\tilde{\omega}_{2a}(t) + \delta\tilde{\omega}_{2b}(t)$  [which should then be given by  $(N_c - 1)\delta\omega_{2a} + (N_c - 1)^2\delta\omega_{2b}$ ]. This quantity, after division by  $(N_c - 1)t^4$ , would behave as what is plotted in Fig. 4 for various  $N_c$ 's. For a given  $N_c \geq 2$  (note that  $N_c = 1$  gives only a non-interacting gas) there would be a  $\ln t$  contribution. The range where this  $\ln t$  would be evident actually decreases with  $N_c$ , and even for  $N_c = 2$  is restricted to  $t < 0.1$ , as compared with  $\sim 0.2$  for  $\omega_{2a}(t)$  in Fig. 2. This is due to the contribution from the “bump” near  $t \sim 0.1$  arising from the  $t^6, t^8, \dots$  contributions we described for  $\delta\omega_{2b}(t)$  for the inset of Fig. 3. However, if data for various  $N_c$ 's are available, one can in principle extrapolate the data at a given  $t$  to  $N_c = 1$  and obtain the non-analytic term  $\delta\omega_{2a}$ . Note that Fig. 4 implies that, at large  $N_c$ , the non-analytic contribution from  $\tilde{\omega}_{2a}$  becomes less and less important as compared with  $\tilde{\omega}_{2b}$ , in accordance with the expectation that at large  $N_c$ , the thermodynamic potential is more and more mean-field like (see Eq. (6)).

### III. DISCUSSIONS

In principle the non-analytic contribution to  $\Omega$  can also be studied for a two-component resonant Fermi gas [26–30], at temperatures above the superfluid transition temperature  $T_c$ . At small and negative scattering length  $a$ , the transition temperature is small and there would still be a temperature range where the  $T^4 \ln T$  term should be observable. The advantage of studying this system is that  $k_\mu a$  can be varied over a wide range, and we can study the higher order contributions in  $k_\mu a$  not analyzed in the present paper, though one has to stay sufficiently above  $T_c$  so that pairing fluctuations would not introduce complications. One can also use the “upper”  $a > 0$  branch

of the Feshbach resonance at magnetic fields where the stability of the gas is not an issue. An advantage of this case is that the higher order interaction terms, not studied in this paper, may give rise to an enhancement for the non-analytic term similar to what occurs in  $^3\text{He}$  [12]. A disadvantage however is that we only have  $N_c = 2$  and the extrapolation procedure described near the end of the last section is not available. The  $T^4 \ln T$  term would also be present for an interacting Fermi gas without  $\text{SU}(N)$  symmetry, with again no extrapolating procedure in the component number feasible.

Experimentally, the density  $n(\mu, T)$  can also be measured. Since  $n(\mu, T) = -\partial\Omega(\mu, T)/\partial\mu$ , it is also non-analytic in  $T$  with a  $T^4 \ln T$  contribution when  $N_c \geq 2$ . The necessary formulas can be straight-forwardly derived from the ones we gave here. They are listed in Appendix B and the non-analytic terms can be extracted by a similar analysis as we discussed in text for  $\Omega(\mu, T)$ .

The extraction of the non-analytic terms in the thermodynamic potential or density seems demanding as very accurate experimental data would be required. On the other hand, these studies would shed valuable new light on an old and interesting problem.

### IV. ACKNOWLEDGEMENT

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### Appendix A: Zero Temperature

Here we verify that Eq. (5), together with Eqs. (7)–(10), does yield the correct result for the energy  $E$  given in the literature [9, 34, 35]. The total number of particles  $N_{\text{tot}}$  can be obtained by  $-\partial\Omega/\partial\mu$ , and so

$$N_{\text{tot}} = VN_c \frac{k_\mu^3}{6\pi^2} \left\{ 1 - 3(k_\mu a)\tilde{\omega}_1 - \frac{7}{2}(k_\mu a)^2\tilde{\omega}_2 \right\} \quad (\text{A1})$$

hence, with  $E = \Omega + \mu N_{\text{tot}}$ ,

$$E = VN_c \frac{k_\mu^3}{6\pi^2} \frac{k_\mu^2}{2M} \left\{ \frac{3}{5} - 2(k_\mu a)\tilde{\omega}_1 - \frac{5}{2}(k_\mu a)^2\tilde{\omega}_2 \right\} \quad (\text{A2})$$

$k_F$  was already obtained in Eq. (11). Inverting that equation, we obtain

$$k_\mu = k_F \left[ 1 + \tilde{\omega}_1(k_F a) + \left( \frac{7}{6}\tilde{\omega}_2 + 3\tilde{\omega}_1^2 \right) (k_F a)^2 \right] \quad (\text{A3})$$

At this stage, we can already verify the dependence of the chemical potential  $\mu$  on  $k_F$ , since  $\mu \equiv k_\mu^2/2M$ . With the help of the zero temperature values of  $\omega_{1,2a,2b}$  in Eqs. (8),(9) and (10), we get

$$\mu = \frac{k_F^2}{2M} \left[ 1 + (N_c - 1) \frac{4}{3\pi} (k_F a) + (N_c - 1) \frac{4}{15} \frac{11 - 2 \ln 2}{\pi^2} (k_F a)^2 \right] \quad (\text{A4})$$

in agreement with, e.g., Ref. [34]. Note that the last term is proportional to  $(N_c - 1)$  and contributions that are  $(N_c - 1)^2$  mutually cancel. Substituting Eq. (A3) into Eq. (A2) and directly using the relation between  $N_{\text{tot}}$  and  $k_F$  gives us

$$\frac{E}{N_{\text{tot}}} = \frac{k_F^2}{2M} \left[ \frac{3}{5} + (N_c - 1) \frac{2}{3\pi} (k_F a) + (N_c - 1) \frac{4}{35} \frac{11 - 2 \ln 2}{\pi^2} (k_F a)^2 \right] \quad (\text{A5})$$

in agreement with Refs. [9, 34, 35].

in text. We have

### Appendix B: particle density $n(\mu, T)$

We give here the low temperature expansion for the density  $n(\mu, T)$ . We write it in a form similar to  $\Omega(\mu, T)$

$$n(\mu, T) = V N_c \frac{k_\mu^3}{6\pi^2} \{ \tilde{\nu}_0 + (k_\mu a) \tilde{\nu}_1 + (k_\mu a)^2 (\tilde{\nu}_{2a} + \tilde{\nu}_{2b}) \} \quad (\text{B1})$$

where

$$\begin{aligned} \tilde{\nu}_0 &= \nu_0 \\ \tilde{\nu}_1 &= (N_c - 1) \nu_1 \\ \tilde{\nu}_{2a} &= (N_c - 1) \nu_{2a} \\ \tilde{\nu}_{2b} &= (N_c - 1)^2 \nu_{2b} \end{aligned} \quad (\text{B2})$$

$$\nu_1(t) = \frac{2}{3\pi} \left[ -3 - \frac{\pi^2}{4} t^2 + \frac{3\pi^4}{80} t^4 + \dots \right] \quad (\text{B4})$$

$$\nu_{2b}(t) = \frac{4}{3\pi^2} \left[ \frac{7}{2} + \frac{5\pi^2}{16} t^2 - \frac{17\pi^4}{3840} t^4 + \dots \right], \quad (\text{B5})$$

with

$$\nu_0(t) = 1 + \frac{\pi^2}{8} t^2 + \frac{7\pi^4}{640} t^4 + \dots, \quad (\text{B3}) \quad \text{and}$$

$$\nu_{2a}(t) = -\frac{7}{2} \omega_{2a}(0) - \frac{3}{2} B_{22} t^2 + \frac{1}{2} B_{23} t^4 \ln t + \left( \frac{B_{24}}{2} + B_{23} \right) t^4 + \dots, \quad (\text{B6})$$

where  $\omega_{2a}(0)$ ,  $B_{22}$ ,  $B_{23}$ ,  $B_{24}$  are the same coefficients

that appeared in text for  $\Omega(\mu, T)$ .

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